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LEBESGUE SUMMABILITY OF DOUBLE TRIGONOMETRIC SERIES(1)

BY

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ABSTRACT. We formulate a definition of symmetric derivatives of *odd* order for functions of two variables. Our definition is based on expanding in a Taylor's series a weighted average of the function about circles.

The definition is applied to derive results on Lebesgue summability for spherically convergent double trigonometric series.

1. Let $f(t)$ be a function defined in a neighborhood of $t_0 \in \mathbf{R}$. Let k be a natural number. We say that f has at t_0 a kth *symmetric derivative* with value a_k if the following holds:

If $k = 2r$ is even,

$$(1.1)\qquad \tfrac{1}{2}\{f(t_0+t)+f(t_0-t)\} = a_0 + \frac{a_2}{2!}t^2 + \cdots + \frac{a_{2r}}{(2r)!}t^{2r} + o(t^{2r})$$

as $t \to 0$.

If $k = 2r+1$ is odd,

$$(1.2)\qquad \tfrac{1}{2}\{f(t_0+t)-f(t_0-t)\} = a_1 t + \frac{a_3}{3!}t^3 + \cdots + \frac{a_{2r+1}}{(2r+1)!}t^{2r+1} + o(t^{2r+1})$$

as $t \to 0$.

If the limit in (1.1) or (1.2) exists only as $t \to 0$ through a set having 0 as a point of density, then we say f has a kth *symmetric approximate derivative* at t_0 equal to a_k.

These definitions may be found in [7]. They have the following applications to termwise integrated trigonometric series. Let T: $\sum_{n\in\mathbf{Z}} c_n e^{in\theta}$ be a trigonometric series in one variable.

THEOREM A. *If $c_n \to 0$ and T converges at θ_0 to s, then the function*

$$F(\theta) = \frac{c_0}{2}\theta^2 - \sum{}' \frac{c_n}{n^2}e^{in\theta}$$

has at θ_0 a second symmetric derivative with value s.

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THEOREM B. *If $c_n \to 0$ and T converges at θ_0 to a finite sum s then the function*

$$L(\theta) = c_0\theta + \sum{}' \frac{c_n}{in} e^{in\theta}$$

has at θ_0 a first symmetric approximate derivative with value s.

We are concerned in this paper with functions of two variables. We denote points of E_2 by $x = (x_1, x_2) = te^{i\theta}$, and we write integral lattice points $n = (n_1, n_2)$. We set $n \cdot x = n_1 x_1 + n_2 x_2$. We denote the Fourier series of a function F by $S[F]$.

Suppose $F(x)$ is defined in a neighborhood of $x_0 \in E_2$. We say that F has at x_0 an rth *generalized Laplacian* equal to s if F is integrable over each circle $|x - x_0| = t$, for t small, and if

$$\frac{1}{2\pi}\int_0^{2\pi} F(x_0 + te^{i\theta})d\theta = a_0 + \frac{a_2}{2!}t^2 + \cdots + \frac{s}{(2^r r!)^2}t^{2r} + o(t^{2r})$$

as $t \to 0$. This definition is due to V. Shapiro [4] and forms a two dimensional analogue of (1.1) for symmetric derivatives of even order. In [3] and [4], it is used to establish two dimensional analogues of Theorem A.

The purpose of this paper is to give a two dimensional analogue of (1.2) for symmetric derivatives of *odd* order, and to apply it to Lebesgue summability for double trigonometric series.

2. We make the following definition. Let

$$\Omega(\theta) = \cos\theta + \sin\theta.$$

Let $F(x)$ be defined in a neighborhood of $x_0 \in E_2$, and suppose that F is integrable on each circle $|x - x_0| = t$, for t small. Let $k = 2r + 1$ be an odd integer.

DEFINITION. F has at x_0 a *generalized symmetric derivative* of order $2r + 1$ with value s if

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} & F(x_0 + te^{i\theta})\Omega(\theta)d\theta \\ &= a_1 t + a_3 t^3 + \cdots + \frac{s}{2^{2r+1} r!\,(r+1)!}t^{2r+1} + o(t^{2r+1}), \end{aligned} \tag{2.1}$$

as $t \to 0$.

If the limit in (2.1) exists only as t tends to 0 through a set E having 0 as a point of density, we will say F has at x_0 a *generalized symmetric approximate derivative* equal to s.

3. The numerical value of the derivative is given by the following result.

THEOREM 1. *Suppose that $F(x)$ and all partial derivatives of F of order*

$\leqslant 2r+2$ *exist and are continuous in a neighborhood of* $x_0 \in E_2$. *Then* F *has at* x_0 *a* $(2r+1)$*th generalized symmetric derivative with value*

$$s = \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)^r F(x_0).$$

PROOF. We may assume $x_0 = 0$. We apply Taylor's theorem. We write

$$F(r,s) = \left.\frac{\partial^{r+s} F}{\partial x_1^r \partial x_2^s}\right|_0.$$

$$F(te^{i\theta}) = \sum_{j=0}^{2r+1} \frac{1}{j!}\left(t\cos\theta\frac{\partial}{\partial x_1} + t\sin\theta\frac{\partial}{\partial x_2}\right)^j F(0)$$
$$+\frac{1}{(2r+2)!}\left(t\cos\theta\frac{\partial}{\partial x_1} + t\sin\theta\frac{\partial}{\partial x_2}\right)^{2r+2} F(\mu e^{i\theta}),$$

for some $\mu \in (0,t)$.

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} F(te^{i\theta})\Omega(\theta)d\theta \\
&= \sum_{j=0}^{2r+1} \frac{t^j}{j!}\frac{1}{2\pi}\int_0^{2\pi}\left(\cos\theta\frac{\partial}{\partial x_1} + \sin\theta\frac{\partial}{\partial x_2}\right)^j F(0)\Omega(\theta)d\theta \\
&\quad + \frac{t^{2r+2}}{(2r+2)!}\frac{1}{2\pi}\int_0^{2\pi}\left(\cos\theta\frac{\partial}{\partial x_1} + \sin\theta\frac{\partial}{\partial x_2}\right)^{2r+2} F(\mu e^{i\theta})\Omega(\theta)d\theta \\
&= \sum_{j=0}^{2r+1} a_j t^j + R_{2r+2},
\end{aligned}\tag{3.1}$$

where

$$a_j = \frac{1}{j!}\frac{1}{2\pi}\int_0^{2\pi}\sum_{m=0}^{j}\binom{j}{m}F(m,j-m)\cdot\cos^m\theta\sin^{j-m}\theta\Omega(\theta)d\theta.$$

Clearly $a_j = 0$ when j is even.

When j is odd,

$$\begin{aligned}
a_j &= \frac{1}{j!}\sum_{m=0}^{j}\binom{j}{m}F(m,j-m)\cdot\frac{1}{2\pi}\int_0^{2\pi}\cos^m\theta\sin^{j-m}\theta\Omega(\theta)d\theta \\
&= \frac{1}{j!}\sum_{m=0}^{j}\binom{j}{m}F(m,j-m)\left\{\frac{1}{2\pi}\int_0^{2\pi}\cos^{m+1}\theta\sin^{j-m}\theta d\theta\right. \\
&\qquad\left. + \frac{1}{2\pi}\int_0^{2\pi}\cos^m\theta\sin^{j-m+1}\theta d\theta\right\} \\
&= \frac{1}{j!}\sum_{m=0}^{j}\binom{j}{m}F(m,j-m)\{c_{jm} + d_{jm}\}.
\end{aligned}\tag{3.2}$$

Using reduction formulae we find,

$$c_{jm} = \begin{cases} \dfrac{m!\,(j-m)!}{2^j((j+1)/2)!\,((m-1)/2)!\,((j-m)/2)!} & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even,} \end{cases}$$

and

$$d_{jm} = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ \dfrac{m!\,(j-m+1)!}{2^{j+1}((j+1)/2)!\,((j-m+1)/2)!\,(m/2)!} & \text{if } m \text{ is even}\,. \end{cases}$$

Breaking the sum in (3.2) into two parts,

(3.3)

$$\begin{aligned} a_j = {} & \sum_{m=0;m\,\text{odd}}^{j} \frac{1}{j!}\binom{j}{m} \frac{m!\,(j-m)!}{2^j((j+1)/2)!\,((m-1)/2)!\,((j-m)/2)!} F(m,j-m) \\ & + \sum_{m=0;m\,\text{even}}^{j} \frac{1}{j!}\binom{j}{m} \frac{m!\,(j-m+1)!}{2^{j+1}((j+1)/2)!\,((j-m+1)/2)!\,(m/2)!} F(m,j-m) \\ = {} & \mathrm{I} + \mathrm{II}. \end{aligned}$$

To simplify I, set $s = (m-1)/2$.

(3.4)

$$\begin{aligned} \mathrm{I} &= \frac{1}{2^j((j+1)/2)!} \sum_{s=0}^{(j-1)/2} \frac{1}{s!\,((j-1)/2-s)!} F(2s+1,j-2s-1) \\ &= \frac{1}{2^j((j+1)/2)!\,((j-1)/2)!} \sum_{s=0}^{(j-1)/2} \binom{(j-1)/2}{s} F(2s+1,j-2s-1) \\ &= \frac{1}{2^j((j+1)/2)!\,((j-1)/2)!} \frac{\partial}{\partial x_1}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)^{(j-1)/2} F(0). \end{aligned}$$

To simplify II, set $s = m/2$.

$$\begin{aligned} \mathrm{II} &= \sum_{m=0;m\,\text{even}}^{j} \frac{j-m+1}{2\cdot 2^j((j+1)/2)!\,((j-m+1)/2)!\,(m/2)!} F(m,j-m) \\ &= \sum_{m=0;m\,\text{even}}^{j} \frac{1}{2^j((j+1)/2)!\,((j-m-1)/2)!\,(m/2)!} F(m,j-m) \\ &= \frac{1}{2^j((j+1)/2)!} \sum_{s=0}^{(j-1)/2} \frac{1}{((j-1)/2-s)!\,s!} F(2s,j-2s) \qquad (3.5) \\ &= \frac{1}{2^j((j+1)/2)!\,((j-1)/2)!} \frac{\partial}{\partial x_2}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)^{(j-1)/2} F(0). \end{aligned}$$

Combining (3.4) and (3.5), we get

(3.6)

$$a_j = \frac{1}{2^j((j+1)/2)!\,((j-1)/2)!}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)^{(j-1)/2} F(0).$$

For the remainder term,

$$R_{2r+2} = t^{2r+2}O(1) = o(t^{2r+1}). \tag{3.7}$$

Substituting (3.6) and (3.7) into (3.1), the proof of Theorem 1 is complete.

4. We now apply the definition in (2.1) to deduce two dimensional versions of Lebesgue summability for spherically convergent double trigonometric series. The role of (2.1) in the extension of Theorem B to two dimensions is parallel to the role played by generalized Laplacians in the extension of Theorem A to two dimensions. Our proof is similar to the methods used in [5], where a different multi-dimensional analogue of Theorem B is given.

THEOREM 2. *Let* T: $\sum_{n\in \mathbf{Z}_2} c_n e^{in\cdot x}$ *be a double trigonometric series which converges spherically at* x_0 *to* s, $s < \infty$. *Suppose the coefficients of* T *satisfy*

$$\sum_{n_1+n_2=0} |n|^{\alpha}|c_n|^2 + \sum_{n_1+n_2\neq 0} |n|^{\alpha}(n_1+n_2)^{-2}|c_n|^2 < \infty, \tag{4.1}$$

for some number $\alpha > 1$. *Then the series*

$$\sum_{n_1+n_2=0} \tfrac{1}{2}(x_1+x_2)c_n e^{in\cdot x} + \sum_{n_1+n_2\neq 0} \frac{-ic_n}{n_1+n_2} e^{in\cdot x} \tag{4.2}$$

converges spherically a.e. on T_2 *to a function* $L(x)$ *which has at* x_0 *a first generalized symmetric approximate derivative equal to* s.

THEOREM 3. *Suppose* $\sum_{n\in\mathbf{Z}_2} c_n e^{in\cdot x}$ *converges spherically at* x_0 *to* s, $s < \infty$. *Suppose there are functions* $L_1(x)$ *and* $L_2(x)$ *such that*

$$\sum_{n_1+n_2=0} c_n e^{in\cdot x} = S[L_1]$$

and

$$\sum_{n_1+n_2\neq 0} \frac{-ic_n}{n_1+n_2} e^{in\cdot x} = S[L_2].$$

Let $L(x) = \frac{1}{2}(x_1+x_2)L_1(x) + L_2(x)$. *Then* $L(x)$ *has at* x_0 *a first generalized symmetric approximate derivative with value* s.

5. Before starting the proofs of Theorems 2 and 3 we establish the following result. Here $J_\nu(z)$ represents the Bessel function of the first kind of order ν.

$$J_\nu(z) = \frac{1}{\pi i^\nu}\int_0^\pi e^{iz\cos\varphi}\cos(\nu\varphi)\,d\varphi.$$

LEMMA. *Let* $x = te^{i\theta} \in E_2$ *and let* $n = (n_1, n_2) \in \mathbf{Z}_2$, *with* $|n| \neq 0$. *Define*

$$g_n(x) = \begin{cases} \dfrac{-ie^{in\cdot x}}{n_1+n_2} & \text{if } n_1+n_2 \neq 0, \\ \frac{1}{2}(x_1+x_2)e^{in\cdot x} & \text{if } n_1+n_2 = 0. \end{cases}$$

Then

$$\frac{1}{2\pi}\int_0^{2\pi} g_n(te^{i\theta})\Omega(\theta)\,d\theta = \frac{J_1(|n|t)}{|n|}.$$

PROOF. Let $n_1/|n| = \cos\varphi$, $n_2/|n| = \sin\varphi$.

We first consider $g_n(x)$ for $n_1+n_2 \neq 0$.

$$\frac{1}{2\pi}\int_0^{2\pi} g_n(te^{i\theta})\Omega(\theta)\,d\theta$$

$$= \frac{|n|}{(n_1+n_2)^2}\cdot\frac{1}{2\pi i}\int_0^{2\pi} \exp\left\{it|n|\left(\frac{n_1}{|n|}\cos\theta + \frac{n_2}{|n|}\sin\theta\right)\right\}$$

$$\cdot(\cos\theta+\sin\theta)\left(\frac{n_1}{|n|}+\frac{n_2}{|n|}\right)d\theta$$

$$= \frac{|n|}{(n_1+n_2)^2}\cdot\frac{1}{2\pi i}\int_0^{2\pi} e^{i|n|t\cos(\theta-\varphi)}(\cos(\theta-\varphi)+\sin(\theta+\varphi))\,d\theta$$

$$= \frac{|n|}{(n_1+n_2)^2}\cdot\frac{1}{2\pi i}\int_0^{2\pi} e^{i|n|t\cos(\theta-\varphi)}\cos(\theta-\varphi)\,d\theta$$

$$+\frac{|n|}{(n_1+n_2)^2}\,\frac{1}{2\pi i}\int_0^{2\pi} e^{i|n|t\cos(\theta-\varphi)}\sin(\theta+\varphi)\,d\theta$$

$$= A_1 + B_1.$$

$$A_1 = \frac{|n|}{(n_1+n_2)^2}J_1(|n|t).$$

Let $\mu = \theta - \varphi$.

$$B_1 = \frac{|n|}{(n_1+n_2)^2}\,\frac{1}{2\pi i}\int_0^{2\pi} e^{i|n|t\cos\mu}\sin(\mu+2\varphi)\,d\mu$$

$$= \frac{|n|}{(n_1+n_2)^2}\cos 2\varphi\frac{1}{2\pi i}\int_0^{2\pi} e^{i|n|t\cos\mu}\sin\mu\,d\mu$$

$$+\frac{|n|}{(n_1+n_2)^2}\sin 2\varphi\frac{1}{2\pi i}\int_0^{2\pi} e^{i|n|t\cos\mu}\cos\mu\,d\mu$$

$$= 0 + \frac{|n|}{(n_1+n_2)^2}\sin(2\varphi)J_1(|n|t) = \frac{|n|}{(n_1+n_2)^2}\,\frac{2n_1n_2}{|n|^2}J_1(|n|t).$$

Combining,

$$\frac{1}{2\pi i}\int_0^{2\pi} g_n(te^{i\theta})\Omega(\theta)\,d\theta$$

$$= A_1 + B_1 = \left(1 + \frac{2n_1 n_2}{|n|^2}\right)\frac{|n|}{(n_1 + n_2)^2}J_1(|n|t) = \frac{J_1(|n|t)}{|n|}.$$

In the case $n_1 + n_2 = 0$,

$$\frac{1}{2\pi}\int_0^{2\pi} g_n(te^{i\theta})\Omega(\theta)\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \tfrac{1}{2}(t\cos\theta + t\sin\theta)e^{in\cdot te^{i\theta}}(\cos\theta + \sin\theta)\,d\theta$$

$$= \frac{t}{4\pi}\int_0^{2\pi} (\cos\theta + \sin\theta)^2 e^{i|n|t\cos(\theta-\varphi)}\,d\theta$$

$$= \frac{t}{4\pi}\int_0^{2\pi} e^{i|n|t\cos(\theta-\varphi)}\,d\theta + \frac{t}{4\pi}\int_0^{2\pi} 2\cos\theta\sin\theta e^{i|n|t\cos(\theta-\varphi)}\,d\theta$$

$$= A_2 + B_2.$$

$$A_2 = \tfrac{1}{2}tJ_0(|n|t).$$

$$B_2 = \frac{t}{4\pi}\int_0^{2\pi} \sin 2(\mu + \varphi)e^{i|n|t\cos\mu}\,d\mu$$

$$= \cos(2\varphi)\frac{t}{4\pi}\int_0^{2\pi} \sin(2\mu)e^{i|n|t\cos\mu}\,d\mu$$

$$+ \sin(2\varphi)\frac{t}{4\pi}\int_0^{2\pi} \cos(2\mu)e^{i|n|t\cos\mu}\,d\mu$$

$$= 0 - \sin(-\pi/2)\tfrac{1}{2}tJ_2(|n|t) = \tfrac{1}{2}tJ_2(|n|t).$$

Combining A_2 and B_2,

$$\frac{1}{2\pi}\int_0^{2\pi} g_n(te^{i\theta})\Omega(\theta)\,d\theta = \tfrac{1}{2}t(J_0(|n|t) + J_2(|n|t)) = \frac{J_1(|n|t)}{|n|}$$

by a formula from [1, p. 12]. Thus the proof of the Lemma is complete.

6. **Proof of Theorem 3.** We will assume, as we may, that $x_0 = 0$ and $s = 0$. We must show

$$\lim_{t\to 0}\text{app}\ \frac{1}{2\pi t}\int_0^{2\pi} L(te^{i\theta})\Omega(\theta)\,d\theta = 0.$$

Set

$$L_1(x,r) = \sum_{n_1+n_2=0} c_n e^{in\cdot x} e^{-|n|r},$$

$$L_2(x,r) = \sum_{n_1+n_2\neq 0} \frac{-ic_n}{n_1+n_2} e^{in\cdot x} e^{-|n|r}$$

and let $L(x,r) = \frac{1}{2}(x_1+x_2)L_1(x,r) + L_2(x,r)$. Using results found in [6], for example, we obtain

$$\begin{aligned}
&\lim_{r\to 0}\int_{T_2} |L(x) - L(x,r)|dx \\
&\qquad \leqslant \lim_{r\to 0}\int_{T_2} |L_1(x) - L_1(x,r)|dx + \lim_{r\to 0}\int_{T_2} |L_2(x) - L_2(x,r)|dx \\
&\qquad = 0.
\end{aligned}$$

Choose a sequence μ_k decreasing to 0 such that

$$\int_{T_2} |L(x) - L(x,\mu_k)|dx \leqslant 2^{-3k-1}.$$

Let

$$C_k = \left\{t \in (0,1) \,\middle|\, \int_0^{2\pi} |L(te^{i\theta}) - L(te^{i\theta},\mu_k)|d\theta > 2^{-k}\right\}.$$

Then

$$\begin{aligned}
2^{-3k-1} &\geqslant \int_0^1 t\,dt \int_0^{2\pi} |L(te^{i\theta}) - L(te^{i\theta},\mu_k)|d\theta \\
&\geqslant \int_{C_k} t2^{-k}\,dt \geqslant \int_0^{|C_k|} t2^{-k}\,dt \\
&= 2^{-k-1}|C_k|^2.
\end{aligned}$$

Hence, $|C_k| \leqslant 2^{-k}$. Thus if we let

$$T = (0,1) - \bigcap_{n=1}^{\infty}\left(\bigcup_{k\geqslant n} C_k\right),$$

then $|T| = 0$ and, outside of T,

$$\lim_{k\to\infty}\int_0^{2\pi} |L(te^{i\theta}) - L(te^{i\theta},\mu_k)|\,d\theta = 0,$$

so $\lim_{k\to\infty}\int_0^{2\pi} |L(te^{i\theta})\Omega(\theta) - L(te^{i\theta},\mu_k)\Omega(\theta)|\,d\theta = 0$. Thus, for almost all $t \in (0,1)$,

$$\lim_{k\to\infty}\frac{1}{2\pi}\int_0^{2\pi} L(te^{i\theta},\mu_k)\Omega(\theta)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} L(te^{i\theta})\Omega(\theta)\,d\theta. \tag{6.1}$$

For $t \in (0,1)$, define

$$\varphi(t) = \lim_{k\to\infty} \frac{1}{2\pi t}\int_0^{2\pi} L(te^{i\theta}, \mu_k)\Omega(\theta)\,d\theta. \tag{6.2}$$

Then, applying the Lemma,

$$\varphi(t) = \lim_{k\to\infty} \frac{1}{2\pi t}\int_0^{2\pi} \lim_{R\to\infty}\left(\sum_{|n|<R} c_n g_n(te^{i\theta})e^{-|n|\mu_k}\right)\cdot\Omega(\theta)\,d\theta$$

$$= \lim_{k\to\infty}\lim_{R\to\infty}\sum_{|n|<R} t^{-1}c_n\cdot\frac{1}{2\pi}\int_0^{2\pi} g_n(te^{i\theta})\Omega(\theta)\,d\theta e^{-|n|\mu_k} \tag{6.3}$$

$$= \lim_{k\to\infty}\lim_{R\to\infty}\sum_{|n|<R} c_n\frac{J_1(|n|t)}{|n|t}e^{-|n|\mu_k}.$$

Let $S_u = \sum_{|n|<u} c_n$. Then, summing by parts,

$$\begin{aligned}&\sum_{|n|<R} c_n\frac{J_1(|n|t)}{|n|t}e^{-|n|\mu_k}\\ &\qquad = -\int_0^R S_u\frac{d}{du}\left(\frac{J_1(ut)}{ut}e^{-u\mu_k}\right)du + S_R\frac{J_1(Rt)}{Rt}e^{-R\mu_k}.\end{aligned} \tag{6.4}$$

Since $S_R = o(1)$ as $R\to\infty$, and using the identity $d(t^{-\nu}J_\nu(t))/dt = -t^{-\nu}J_{\nu+1}(t)$, we get

$$S_R\frac{J_1(Rt)}{Rt}e^{-R\mu_k}\to 0$$

as $R\to\infty$. Hence the last term on the right side of (6.4) drops out, and

$$\begin{aligned}&\lim_{R\to\infty}\sum_{|n|<R} c_n\frac{J_1(|n|t)}{|n|t}e^{-|n|\mu_k}\\ &\qquad = -\int_0^\infty S_u\frac{d}{du}\left(\frac{J_1(ut)}{ut}e^{-u\mu_k}\right)du\\ &\qquad = -\int_0^\infty S_u\left\{\frac{J_2(ut)}{u}e^{-u\mu_k} - \mu_k\frac{J_1(ut)}{ut}e^{-u\mu_k}\right\}du.\end{aligned}$$

Returning to (6.3),

$$\begin{aligned}\varphi(t) &= -\lim_{k\to\infty}\int_0^\infty S_u\frac{J_2(ut)}{u}e^{-u\mu_k}du + \lim_{k\to\infty}\mu_k\int_0^\infty S_u\frac{J_1(ut)}{ut}e^{-u\mu_k}du\\ &= -\lim_{k\to\infty}\int_0^\infty S_u\frac{J_2(ut)}{u}e^{-u\mu_k}du.\end{aligned}$$

We claim

$$\int_\rho^{2\rho}|\varphi(t)|\,dt = o(\rho)\quad\text{as } \rho\to 0. \tag{6.5}$$

For,

$$\int_\rho^{2\rho} |\varphi(t)|\,dt = \int_\rho^{2\rho} \left| \lim_{k\to\infty} \int_0^\infty S_u \frac{J_2(ut)}{u} e^{-u\mu_k} du \right| dt$$

$$\leqslant \int_\rho^{2\rho} \int_0^\infty \left| S_u \frac{J_2(ut)}{u} \right| du\,dt = \int_0^\infty \int_\rho^{2\rho} \left| S_u \frac{J_2(ut)}{u} \right| dt\,du$$

$$= \int_0^{1/\rho} \int_\rho^{2\rho} \left| S_u \frac{J_2(ut)}{u} \right| dt\,du + \int_{1/\rho}^\infty \int_\rho^{2\rho} \left| S_u \frac{J_2(ut)}{u} \right| dt\,du$$

$$= P + Q.$$

We use the relations $|J_\nu(t)| \leqslant ct^\nu$ for $0 < t \leqslant 2$, and $|J_\nu(t)| \leqslant c't^{-1/2}$ for $t > 1$.

In the interval of integration involving P, $|ut| \leqslant 2$, so $|u^{-1}J_2(ut)| \leqslant cut^2$.

$$P = \int_0^{1/\rho} \int_\rho^{2\rho} o(1) O(ut^2)\,dt\,du = o(\rho).$$

In the interval of integration for Q, $ut > 1$, so $|J_2(ut)| \leqslant c(ut)^{-1/2}$.

$$Q = \int_{1/\rho}^\infty \int_\rho^{2\rho} o(1) u^{-1} O(ut)^{-1/2} dt\,du = o(\rho).$$

Thus the claim is established.

We complete the proof of Theorem 2 as follows. Let

$$\int_{2^{-n-1}}^{2^{-n}} |\varphi(t)|\,dt = 2^{-n}\varepsilon_n, \tag{6.6}$$

where $\varepsilon_n \to 0$ as $n \to \infty$. Let $E_n = \{t \in [2^{-n-1}, 2^{-n}]\colon |\varphi(t)| > \sqrt{\varepsilon_n}\}$. Then

$$\int_{2^{-n-1}}^{2^{-n}} |\varphi(t)|\,dt \geqslant |E_n|\sqrt{\varepsilon_n},$$

so using (6.6), $2^{-n}\varepsilon_n \geqslant \sqrt{\varepsilon_n}|E_n|$, and $|E_n| \leqslant 2^{-n}\sqrt{\varepsilon_n}$. Now let $E = T - \bigcup_{n=1}^\infty E_n$. Then E has 0 as a point of density. In E, $\varphi(t) \to 0$, and $\varphi(t) = 1/(2\pi t) \int_0^{2\pi} L(te^{i\theta})\Omega(\theta)\,d\theta$. Thus, the theorem is established.

7. **Proof of Theorem 2.** Let

$$T_R(x) = \sum_{|n|<R;\, n_1+n_2=0} \tfrac{1}{2}(x_1 + x_2) c_n e^{in\cdot x} + \sum_{|n|<R;\, n_1+n_2\neq 0} \frac{-ic_n}{n_1 + n_2} e^{in\cdot x}.$$

The condition (4.1) insures that $L(x) = \lim_{R\to\infty} T_R(x)$ exists a.e. on each circle $|x| = t$. This is a consequence of Theorem 1 of [2]. Moreover, by Theorem 2 of [2], $\int_0^{2\pi} \sup_R |T_R(te^{i\theta})|\,d\theta < \infty$, so

$$\frac{1}{2\pi t}\int_0^{2\pi} L(te^{i\theta})\Omega(\theta)\,d\theta = \lim_{R\to\infty} \frac{1}{2\pi t}\int_0^{2\pi} T_R(te^{i\theta})\Omega(\theta)\,d\theta$$
$$= \lim_{R\to\infty} \sum_{|n|<R} c_n \cdot \frac{1}{2\pi t}\int_0^{2\pi} g_n(te^{i\theta})\Omega(\theta)\,d\theta$$
$$= \lim_{R\to\infty} \sum_{|n|<R} c_n \frac{J_1(|n|t)}{|n|t}.$$

We now let

$$\varphi(t) = \frac{1}{2\pi t}\int_0^{2\pi} L(te^{i\theta})\Omega(\theta)\,d\theta.$$

Summing by parts,

$$\varphi(t) = \int_0^\infty S_u \frac{J_2(ut)}{u}\,du.$$

The verification of the claim (6.5) and the completion of the proof follow exactly the lines of the completion of the proof of Theorem 3.

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